113 Class Problems: Polynomials and Fields of Fractions

1. (a) Find all the units in $\mathbb{Z}/5\mathbb{Z}[x]$. What do you notice about them?

   (b) Give an example of a non-constant polynomial in $\mathbb{Z}/4\mathbb{Z}[x]$ which is a unit. What is the difference between this case and part (a).

   Solutions:

a) $\mathbb{Z}/5\mathbb{Z}$ a field $\Rightarrow$ $\mathbb{Z}/5\mathbb{Z}[x]$ I.D. Hence if $f(x), g(x) \neq 0$

$f(x)g(x) \neq 0$ and $\deg(f(x)g(x)) = \deg(g(x)) + \deg(f(x))$

Hence $f(x) \in (\mathbb{Z}/5\mathbb{Z}[x])^* \Rightarrow f(x) \in \mathbb{Z}/5\mathbb{Z}^*$

$\Rightarrow f(x) = [1], [2], [3], [4]$ "constant" polynomials

b) $([1]+[2]x)([1]+[2]x) = [1] \Rightarrow [1]+[2]x \in (\mathbb{Z}/4\mathbb{Z}[x])^*$

2. Let R be an integral domain. Prove the transitivity property of $\sim$ on $R\times(R\backslash\{0_R\})$. Give an example to show that if R is not an integral domain then the transitivity property may not hold.

   Solutions:

$(a,b) \sim (c,d) \iff ad - bc = 0_R$

$(c,d) \sim (e,f) \iff cf - de = 0_R$

$\Rightarrow f\underbrace{(ad-bc)}_{0_R} + b\underbrace{(cf-de)}_{0_R} = d(af - be) = 0_R$

$d \neq 0_R \Rightarrow af - be = 0_R \Rightarrow (a,b) \sim (e,f)$

$R = \mathbb{Z}/4\mathbb{Z}$

$(1,1) \sim (2,2)$

$(2,2) \sim (1,3)$

$(1,1) \not\sim (1,3)$

3. Prove that $\mathbb{C}$ is isomorphic (as a ring) to a quotient ring of $\mathbb{R}[x]$. Hint: Use the first isomorphism theorem

Solutions:

Define $\phi : \mathbb{R}[x] \longrightarrow \mathbb{C}$

$f(x) \longmapsto f(i)$

$\mathbb{C}$ commutative $\Rightarrow \phi$ a homomorphism

First isomorphism theorem

$\operatorname{Im} \phi = \mathbb{C} \Rightarrow \mathbb{R}[x] / \operatorname{Ker} \phi \cong \mathbb{C}$